

A Quantum Generalized Mittag-Leffler Function Via Caputo q-Fractional Equations

Thabet Abdeljawad and Betül Benli
Department of Mathematics and Computer Science
Çankaya University, 06530 Ankara, Turkey

January 12, 2013

Abstract

Some Caputo q-fractional difference equations are solved. The solutions are expressed by means of a new introduced generalized type of q-Mittag-Leffler functions. The method of successive approximation is used to obtain the solutions. The obtained q-version of Mittag-Leffler function is thought as the q-analogue of the one introduced previously by Kilbas and Saigo.

AMS Subject Classification: 26A33; 60G05; 60G07; 60G012; 60GH05, 41A05, 33D60, 34G10.

Key Words and Phrases: q-fractional integral, Caputo q-fractional derivatives, generalized q-Mittag-Leffler function.

1 Introduction and Preliminaries

The concept of fractional calculus is not new. However, it has gained its popularity and importance during the last three decades or so. This is due to its distinguished applications in numerous diverse fields of science and engineering ([16], [15], [17]). The q-calculus is also not of recent appearance. It was initiated in twenties of the last century. For the basic concepts in q-calculus we refer the reader to [9]. Starting from the q-analogue of Cauchy formula [13], Al-Salam started the fitting of the concept of q-fractional calculus. After that he ([12], [11]) and Agarwal R. [10] continued on by studying certain q-fractional integrals and derivatives, where they proved the semigroup properties for left and right (Riemann) type fractional integrals but without variable lower limit and variable upper limit, respectively. Recently, the authors in [14] generalized the notion of the (left) fractional q-integral and q-derivative by introducing variable lower limit and proved the semigroup properties.

Very recently and after the appearance of time scale calculus (see for example [7]), some authors started to pay attention and apply the techniques of time

scale to discrete fractional calculus ([4],[5],[6], [2]) benefitting from the results announced before in [8]. All of these results are mainly about fractional calculus on the time scales $T_q = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$ and $h\mathbb{Z}$ [3]. As a contribution in this direction and being motivated by all above, in this article we introduce the q-analogue of a generalized type Mittag-Leffler function used before by Kilbas and Saigo in [18]. Such functions are obtained by solving linear q-Caputo initial value problems. The results obtained in this article generalize also the results of [1].

For the theory of q-calculus we refer the reader to the survey [9] and for the basic definitions and results for the q-fractional calculus we refer to [6]. Here we shall summarize some of those basics.

For $0 < q < 1$, let T_q be the time scale

$$T_q = \{q^n : n \in \mathbb{Z}\} \cup \{0\}.$$

where \mathbb{Z} is the set of integers. More generally, if α is a nonnegative real number then we define the time scale

$$T_q^\alpha = \{q^{n+\alpha} : n \in \mathbb{Z}\} \cup \{0\},$$

we write $T_q^0 = T_q$.

For a function $f : T_q \rightarrow \mathbb{R}$, the nabla q-derivative of f is given by

$$\nabla_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \in T_q - \{0\} \quad (1)$$

The nabla q-integral of f is given by

$$\int_0^t f(s) \nabla_q s = (1-q)t \sum_{i=0}^{\infty} q^i f(tq^i) \quad (2)$$

and for $0 \leq a \in T_q$

$$\int_a^t f(s) \nabla_q s = \int_0^t f(s) \nabla_q s - \int_0^a f(s) \nabla_q s$$

On the other hand

$$\int_t^{\infty} f(s) \nabla_q s = (1-q)t \sum_{i=1}^{\infty} q^{-i} f(tq^{-i}) \quad (3)$$

and for $0 < b < \infty$ in T_q

$$\int_t^b f(s) \nabla_q s = \int_t^{\infty} f(s) \nabla_q s - \int_b^{\infty} f(s) \nabla_q s \quad (4)$$

By the fundamental theorem in q-calculus we have

$$\nabla_q \int_0^t f(s) \nabla_q s = f(t) \quad (5)$$

and if f is continuous at 0, then

$$\int_0^t \nabla_q f(s) \nabla_q s = f(t) - f(0) \quad (6)$$

Also the following identity will be helpful

$$\nabla_q \int_a^t f(t, s) \nabla_q s = \int_a^t \nabla_q f(t, s) \nabla_q s + f(qt, t) \quad (7)$$

Similarly the following identity will be useful as well

$$\nabla_q \int_t^b f(t, s) \nabla_q s = \int_{qt}^b \nabla_q f(t, s) \nabla_q s - f(t, t) \quad (8)$$

The q-derivative in (7) and (8) is applied with respect to t.

From the theory of q-calculus and the theory of time scale more generally, the following product rule is valid

$$\nabla_q(f(t)g(t)) = f(qt)\nabla_q g(t) + \nabla_q f(t)g(t) \quad (9)$$

The q-factorial function for $n \in \mathbb{N}$ is defined by

$$(t-s)_q^n = \prod_{i=0}^{n-1} (t - q^i s) \quad (10)$$

When α is a non positive integer, the q-factorial function is defined by

$$(t-s)_q^\alpha = t^\alpha \prod_{i=0}^{\infty} \frac{1 - \frac{s}{t} q^i}{1 - \frac{s}{t} q^{i+\alpha}} \quad (11)$$

We summarize some of the properties of q-factorial functions, which can be found mainly in [6], in the following lemma

Lemma 1. (i) $(t-s)_q^{\beta+\gamma} = (t-s)_q^\beta (t-q^\beta s)_q^\gamma$
(ii) $(at-as)_q^\beta = a^\beta (t-s)_q^\beta$
(iii) The nabla q-derivative of the q-factorial function with respect to t is

$$\nabla_q (t-s)_q^\alpha = \frac{1-q^\alpha}{1-q} (t-s)_q^{\alpha-1}$$

(iv) The nabla q-derivative of the q-factorial function with respect to s is

$$\nabla_q (t-s)_q^\alpha = -\frac{1-q^\alpha}{1-q} (t-qs)_q^{\alpha-1}$$

where $\alpha, \gamma, \beta \in \mathbb{R}$.

Definition 2. [1] Let $\alpha > 0$. If $\alpha \notin \mathbb{N}$, then the α -order Caputo (left) q -fractional derivative of a function f is defined by

$${}_q C_a^\alpha f(t) \triangleq {}_q I_a^{(n-\alpha)} \nabla_q^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-qs)_q^{n-\alpha-1} \nabla_q^n f(s) \nabla_q s \quad (12)$$

where $n = [\alpha] + 1$.

If $\alpha \in \mathbb{N}$, then ${}_q C_a^\alpha f(t) \triangleq \nabla_q^n f(t)$

It is clear that ${}_q C_a^\alpha$ maps functions defined on T_q to functions defined on T_q , and that ${}_b C_q^\alpha$ maps functions defined on $T_q^{1-\alpha}$ to functions defined on T_q

The following identity which is useful to transform Caputo q -fractional difference equations into q -fractional integrals, will be our key in solving the q -fractional linear type equation by using successive approximation.

Proposition 3. [1] Assume $\alpha > 0$ and f is defined in suitable domains. Then

$${}_q I_a^\alpha {}_q C_a^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)_q^k}{\Gamma_q(k+1)} \nabla_q^k f(a) \quad (13)$$

and if $0 < \alpha \leq 1$ then

$${}_q I_a^\alpha {}_q C_a^\alpha f(t) = f(t) - f(a) \quad (14)$$

The following identity [14] is essential to solve linear q -fractional equations

$${}_q I_a^\alpha (x-a)_q^\mu = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\alpha+\mu+1)} (x-a)_q^{\mu+\alpha} \quad (0 < a < x < b) \quad (15)$$

where $\alpha \in \mathbb{R}^+$ and $\mu \in (-1, \infty)$. The q -analogue of Mittag-Leffler function with double index (α, β) is introduced in [1]. It was defined as follows:

Definition 4. [1] For $z, z_0 \in \mathbb{C}$ and $\Re(\alpha) > 0$, the q -Mittag-Leffler function is defined by

$${}_q E_{\alpha, \beta}(\lambda, z - z_0) = \sum_{k=0}^{\infty} \lambda^k \frac{(z - z_0)_q^{\alpha k}}{\Gamma_q(\alpha k + \beta)}. \quad (16)$$

When $\beta = 1$ we simply use ${}_q E_\alpha(\lambda, z - z_0) := {}_q E_{\alpha, 1}(\lambda, z - z_0)$.

2 Main Results

The following is to be the q -analogue of the generalized Mittag-Leffler function introduced by Kilbas and Saigo [18] (see also [17] page 48).

Definition 5. For $\alpha, l, \lambda \in \mathbb{C}$ are complex numbers and $m \in \mathbb{R}$ such that $\Re(\alpha) > 0$, $m > 0$, $a \geq 0$ and $\alpha(jm + l) \neq -1, -2, -3, \dots$, the generalized q-Mittag-Leffler function (of order 0) is defined by

$${}_qE_{\alpha, m, l}(\lambda, x - a) = 1 + \sum_{k=1}^{\infty} \lambda^k q^{-\frac{k(k-1)}{2}\alpha(m-1)(\alpha l + \alpha)} c_k (x - a)_q^{\alpha k m}$$

where

$$c_k = \prod_{j=0}^{k-1} \frac{\Gamma_q[\alpha(jm + l) + 1]}{\Gamma_q[\alpha(jm + l + 1) + 1]}, \quad k = 1, 2, 3, \dots$$

While the the generalized q-Mittag-Leffler function (of order r), $r = 0, 1, 2, 3, \dots$, is defined by

$${}_qE_{\alpha, m, l}^r(\lambda, x - a) = 1 + \sum_{k=1}^{\infty} \lambda^k q^{-k\alpha(m-1)r} q^{-\frac{k(k-1)}{2}\alpha(m-1)(\alpha l + \alpha)} c_k (x - q^r a)_q^{\alpha k m}.$$

Note that ${}_qE_{\alpha, m, l}^0(\lambda, x - a) = {}_qE_{\alpha, m, l}(\lambda, x - a)$.

Remark 6. In particular, if $m = 1$, then the generalized q-Mittag-Leffler function is reduced to the q-Mittag-Leffler function, apart from a constant factor $\Gamma_q(\alpha l + 1)$. Namely,

$${}_qE_{\alpha, 1, l}(\lambda, x - a) = \Gamma_q(\alpha l + 1) {}_qE_{\alpha, \alpha l + 1}(\lambda, x - a) \quad (17)$$

This turns to be the q-analogue of the identity $E_{\alpha, 1, l}(z) = \Gamma(\alpha l + 1) E_{\alpha, \alpha l + 1}(z)$ (see [17]) page 48).

Example 7. Consider the q-Caputo difference equation

$$({}_qC_a^\alpha y)(x) = \lambda(x - a)_q^\beta y(q^{-\beta} x), \quad y(a) = b \quad (18)$$

where

$$0 < \alpha < 1, \quad \beta > -\alpha, \quad \lambda \in \mathbb{R}, \quad b \in \mathbb{R}.$$

Applying Proposition 3 we have

$$y(x) = y(a) + \lambda {}_qI_a^\alpha [(x - a)_q^\beta y(q^{-\beta} x)].$$

The method of successive applications implies that

$$y_m(x) = y(a) + \lambda {}_qI_a^\alpha [(x - a)_q^\beta y_{m-1}(q^{-\beta} x)], \quad m = 1, 2, 3, \dots,$$

where $y_0(x) = b$. Then by the help of (15) we have

$$y_1(x) = b + b\lambda \frac{\Gamma_q(\beta + 1)}{\Gamma_q(\beta + \alpha + 1)} (x - a)_q^{\beta + \alpha},$$

and

$$y_2(x) = b + b\lambda {}_qI_a^\alpha [(x-a)_q^\beta \{1 + \lambda \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)} (q^{-\beta}x - a)_q^{\beta+\alpha}\}]$$

Then by (i) and (ii) of Lemma 1

$$y_2(x) = b + b\lambda {}_qI_a^\alpha [(x-a)_q^\beta + \lambda \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)} q^{-\beta(\alpha+\beta)} (x-a)_q^{2\beta+\alpha}]$$

Again by (15) we conclude

$$y_2(x) = b + b\lambda {}_qI_a^\alpha [(x-a)_q^\beta + \lambda \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)} q^{-\beta(\alpha+\beta)} (x-a)_q^{2\beta+\alpha}]$$

Then (15) leads to

$$y_2(x) = b[1 + \lambda \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)} (x-a)_q^{\beta+\alpha} + \lambda^2 \frac{\Gamma_q(2\beta+\alpha+1)}{\Gamma_q(2\beta+2\alpha+1)} q^{-\beta(\alpha+\beta)} (x-a)_q^{2\beta+2\alpha}]. \quad (19)$$

Proceeding inductively, for each $m = 1, 2, \dots$ we obtain

$$y_m(x) = b[1 + \sum_{k=1}^m \lambda^k q^{-\beta \frac{k(k-1)}{2}(\alpha+\beta)} c_k (x-a)_q^{k(\alpha+\beta)}] \quad (20)$$

where

$$c_k = \prod_{j=0}^{k-1} \frac{\Gamma_q[\alpha(jm+l)+1]}{\Gamma_q[\alpha(jm+l+1)+1]}, \quad m = 1 + \frac{\beta}{\alpha}, \quad l = \frac{\beta}{\alpha}, \quad k = 1, 2, 3, \dots$$

If we let $m \rightarrow \infty$, then we obtain the solution

$$y(x) = b [1 + \sum_{k=1}^{\infty} \lambda^k q^{-\beta \frac{k(k-1)}{2}(\alpha+\beta)} c_k (x-a)_q^{k(\alpha+\beta)}]$$

which is exactly

$$y(x) = b {}_qE_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}(\lambda, x-a).$$

Remark 8. 1) If in (18) $\beta = 0$, then in accordance with (17) and Example 10 in [1] we have

$${}_qE_{\alpha, 1, 0}(\lambda, x-a) = {}_qE_{\alpha, 1}(\lambda, x-a) = {}_qE_{\alpha}(\lambda, x-a)$$

2) The solution of the q -Cauchy problem

$$({}_qC_a^{\frac{1}{2}}y)(x) = \lambda(x-a)_q^\beta y(q^{-\beta}x), \quad y(a) = b \quad (21)$$

where

$$0 < \alpha < 1, \beta > -\frac{1}{2}, \lambda \in \mathbb{R}, b \in \mathbb{R}$$

is given by

$$y(x) = b {}_qE_{\frac{1}{2}, 1+2\beta, 2\beta}(\lambda, x-a).$$

3) By the help of (13) and Lemma 1 and by applying the successive approximation with $y_0(x) = \sum_{k=0}^{n-1} \frac{(t-a)_q^k}{\Gamma_q(k+1)} \nabla_q^k f(a)$, Example 7 can be generalized for arbitrary $\alpha > 0$. Namely, the solution of the q -initial value problem

$$({}_qC_a^\alpha y)(x) = \lambda(x-a)_q^\beta y(q^{-\beta}x), y^{(k)}(a) = b_k \quad (b_k \in \mathbb{R}, k = 0, 1, \dots, n-1) \quad (22)$$

where

$$n-1 < \alpha < n, \beta > -\alpha, \lambda \in \mathbb{R}, b \in \mathbb{R}$$

is given by

$$y(x) = \sum_{r=0}^{n-1} \frac{b_r}{\Gamma_q(r+1)} (x-a)_q^r {}_qE_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta+r}{\alpha}}^r(\lambda, x-a).$$

Note that when $0 < \alpha < 1$, i.e, $n = 1$, the solution of Example 7 is recovered.

References

- [1] Abdeljawad T. and Dumitru B., Caputo q -Fractional Initial Value Problems and a q -Analogue Mittag-Leffler Function, Communications in Nonlinear Science and Numerical Simulations, to appear.
- [2] Abdeljawad T. and Dumitru B. Fractional differences and integration by parts, Journal of computational Analysis and Applications, vol 13, no. 3, 574-582. .
- [3] Nuno R. O. Bastos, Rui A. C. Ferreira, Delfim F. M. Torres, Discrete-Time Variational Problems, Signal Processing, Volume 91 Issue 3, March, 2011.
- [4] Atıcı F.M. and Elloe P. W., A Transform method in discrete fractional calculus , *International Journal of Difference Equations*, vol 2, no 2, (2007), 165–176.
- [5] Atıcı F.M. and Elloe P. W., Initial value problems in discrete fractional calculus, *Proceedings of the American Mathematical Society*, to appear.
- [6] Atıcı F.M. and Elloe P. W., Fractional q -calculus on a time scale , *Journal of Nonlinear Mathematical Physics* 14, 3, (2007), 333–344.

- [7] Bohner M. and A. Peterson, Dynamic equations on Time Scales, Birkhäuser, Boston, 2001.
- [8] Miller K. S. and Ross B., Fractional difference calculus, *Proceedings of the International Symposium on Univalent Functions, Fractional Calculus and Their Applications*, Nihon University, Koriyama, Japan, (1989), 139-152.
- [9] Ernst T., The history of q-calculus and new method (Licentiate Thesis), U.U.D.M. Report 2000: <http://math.uu.se/thomas/Lics.pdf>.
- [10] Agrawal R. P., Certain fractional q-integrals and q-derivatives, Proc. Camb. Phil. Soc. (1969), 66, 365, 365-370.
- [11] Al-Salam W. A., Some fractional q-integrals and q-derivatives, Proc. Edin. Math. Soc., vol 15 (1969), 135-140.
- [12] Al-Salam W. A. and Verma A., A fractional Leibniz q-formula, Pacific Journal of Mathematics, vol 60, (1975), 1-9.
- [13] Al-Salam W. A., q-Analogues of Cauchy's formula, Proc. Amer. Math. Soc. 17, 182-184, (1952-1953).
- [14] Predrag M.R., Sladana D. M. and Miomir S. S., Fractional Integrals and Derivatives in q-calculus, Applicable Analysis and Discrete Mathematics, 1, 311-323, (2007).
- [15] Podlubny I., Fractional Differential Equations, Academic Press, 1999.
- [16] Samko G. Kilbas A. A., Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Yverdon, 1993.
- [17] Kilbas A., Srivastava M. H., and Trujillo J. J. Theory and Application of Fractional Differential Equations, North Holland Mathematics Studies 204, 2006.
- [18] Kilbas, A. A. and Saigo M., On solution of integral equation of Abel-Volterra type, *Diff. Integr. Equat.*, 8 (5), (1995) 993-1011.